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Twistors and Supersymmetry

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Abstract

An overview is given of the application of twistor geometric ideas to supersymmetry with particular emphasis on the construction of superspaces associated with four-dimensional spacetime.

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1 Introduction

Twistor geometry, suitably interpreted, is particularly well-suited to the description of supersymmetric theories in the superspace formalism. There have been applications: to field theories, notably supersymmetric Yang-Mills (SYM) and supergravity (SG), leading to off-shell representations and to the interpretation of on-shell constraints as integrability conditions, and to super-particles, strings and extended objects in the context of doubly supersymmetric formulations, that is, formulations with both spacetime (target space) and worldsurface supersymmetry.

The term “twistor geometry” as I shall interpret it in the supersymmetric context embraces the three main geometrical approaches to supersymmetry that have been studied: chiral supergeometry, light-like integrability or super twistor theory, and harmonic superspace, each of these three formalisms being particular examples of twistor supergeometries as I shall show. In general one can say that the twistor approach is a very valuable tool for constructing and understanding superspace geometries and, in particular, it clarifies the geometrical structure of harmonic superspace. Since (almost) all supersymmetric theories are amenable to one or more of the above descriptions it follows that twistor supergeometry is universal: it underlies (almost) all supersymmetric theories of interest.

In this talk I shall focus on the construction of various (flat-space) supergeometries associated with four-dimensional spacetime. In four dimensions twistors are naturally associated with conformal symmetry, a symmetry which, as is well-known (see for example [1]), is particularly relevant in the supersymmetric case for three reasons: firstly, massless, non-gravitational theories of interest are classically superconformal, secondly, it is known that there are four-dimensional superconformal quantum field theories, and thirdly, the construction and study of supergravity theories is made much easier if one uses the superconformal perspective. For the most part I shall work in complex spacetime; this has some advantages from a formal point of view and has some relevance in that quantum field theories can be studied in regions of products of several copies of complex spacetime. Moreover, there is no loss of generality since one can easily impose reality in the formalism. In the case of standard twistor theory, there is still a complex twistor space associated with real (Euclidean) space; in the case of harmonic supergeometry one finds that the complex geometry of twistor space has to be replaced by CR supergeometry, which will be briefly explained in section 4. Finally, I shall give a list of some applications to supersymmetric field theories.

2 Twistors

Consider the equation

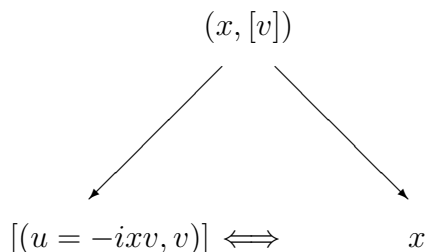
$$u^\alpha = -ix^{\alpha\dot{\alpha}}v_{\dot{\alpha}} \tag{1}$$

where u and v are two-component commuting spinors and x is a four-vector in 2×2 spinor form. The pair $z = (u, v)$ is an element of twistor space $\mathbb{T} \cong \mathbb{C}^4$, and x labels a point in complex Minkowski space $\mathbb{M} \cong \mathbb{C}^4$. Equation (1) establishes a correspondence between

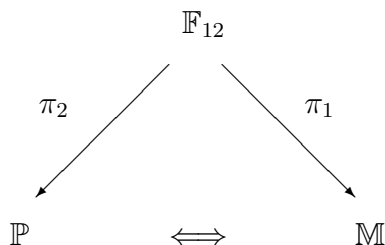
\mathbb{T} and \mathbb{M} in the following way [2]: firstly, if $z = (u, v)$ is held fixed, (1) defines a 2-plane (called a β -plane) in \mathbb{M} which is totally null (all tangent vectors are null) and anti-self-dual (the bivector constructed from any two independent tangents is anti-self-dual); secondly if x is held fixed, then one can solve (1) for u as a function of v . Since (1) is clearly invariant under common rescalings of u and v this second point of view determines a projective line (\mathbb{CP}^1) in projective twistor space, \mathbb{P} , also known as a twistor line. Thus we have the correspondence

- points $x \in M \rightarrow$ twistor lines $\underline{x} \in \mathbb{P}$
- points $[z] \in \mathbb{P} \rightarrow \beta$ -planes $\underline{z} \in \mathbb{M}$

We can present this a little differently as follows:



That is, given a point $x \in \mathbb{M}$ and a projective dotted spinor $[v]$ we can project either onto Minkowski space or onto projective twistor space as indicated. We can rewrite this in terms of spaces as:



where \mathbb{F}_{12} , the space of points of \mathbb{M} and projective dotted spinors, is called the correspondence space and the projections π_1 and π_2 are given in coordinates in the previous diagram. This type of diagram is called a double fibration [4], and the projections should be such that π_2 is one-to-one on each fibre $\pi_1^{-1}(x)$ of π_1 and vice-versa, so that the subset $\underline{x} = \pi_2 \circ \pi_1^{-1}(x)$ in \mathbb{P} is a copy of the fibre $\pi_1^{-1}(x)$ and similarly for $\underline{z} = \pi_1 \circ \pi_2^{-1}(z)$, which is a subset of \mathbb{M} . Thus the double fibration builds in the correspondence automatically. We shall interpret a set of three spaces related to each other by such a double fibration as a twistor geometry.

The significance of such a geometrical set-up lies in the fact that information about field theories on spacetime can be related to certain (holomorphic) data on twistor space. Most importantly, one has the Ward construction for gauge theories [3, 4]. Given a gauge theory on \mathbb{M} we can lift it to \mathbb{F}_{12} to get a gauge theory on the bigger space which depends trivially on the fibre coordinate, $[v]$. Similarly, a gauge theory on \mathbb{P} will lift to a theory on \mathbb{F}_{12} which is trivial on the fibres of π_2 . Thus, in order for gauge theories on \mathbb{M} and \mathbb{P} to be equivalent, the former must be trivial on each fibre of π_2 , i.e. on each β -plane, while the latter must be trivial on each fibre of π_1 , i.e. on each twistor line. More precisely, one finds that a gauge theory on \mathbb{M} with vanishing curvature on each β -plane is equivalent to a holomorphic vector bundle over \mathbb{P} which is trivial on each twistor line. Moreover, this result generalises to any double fibration of the above type provided certain technical requirements are met [5].

In gravity, the situation is slightly different. Given a (complex) spacetime \mathbb{M} one wishes to know if there is a twistor space such that one can construct a double fibration. The existence of such a twistor space leads to constraints on the geometry of \mathbb{M} ; in standard twistor theory these constraints involve self-duality whereas in supergravity one can find generalised twistor spaces whose existence implies the desired equations of constraint on the geometry of superspace.

There is a systematic way of constructing double fibrations using group theory [5]. Regarding twistor space \mathbb{T} as a representation space for the complex conformal group, $SL(4, \mathbb{C})$ we can construct from it a number of spaces (7 in all) which have the following properties:

- They are spaces of flags in \mathbb{T} .
- They are homogeneous spaces of the form $P \backslash G$ where P is a parabolic subgroup of $G := SL(4, \mathbb{C})$
- They group together in sets of 3, since $P_1 \cap P_2 := P_{12}$ is parabolic whenever P_1 and P_2 are, so that one automatically gets double fibrations, $P_2 \backslash G \leftarrow P_{12} \backslash G \rightarrow P_1 \backslash G$, with the desired properties.

Consider the following subgroups of G consisting of matrices of the type indicated (the crosses denote non-zero elements):

$$P_1 = \begin{pmatrix} \times & \times & & \\ \times & \times & & \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \quad P_2 = \begin{pmatrix} \times & & & \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

If we view these matrices as acting from the left on a column vector of basis vectors of \mathbb{T} we see that P_1 leaves the plane determined by the first two vectors invariant, whereas P_2 leaves the line delivered by the first vector invariant. Hence $P_1 \backslash G = \mathbb{G}r_2(4)$, the

Grassmanian of 2-planes in \mathbb{C}^4 while $P_2 \backslash G = \mathbb{CP}^3$. Clearly:

$$P_{12} = \begin{pmatrix} \times & & & \\ \times & \times & & \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

and the space $P_{12} \backslash G$ is the space of flags of type (1,2) in \mathbb{C}^4 , where a flag of type (1,2) is a line embedded in a two-dimensional subspace.

All the above spaces are in fact compact complex manifolds. In particular we can identify $Gr_2(4)$ as complexified compactified Minkowski space $\widetilde{\mathbb{M}}$. In practice we are interested in non-compact spacetime, and we can identify this as an open set homeomorphic to \mathbb{C}^4 in $\widetilde{\mathbb{M}}$. This open set can be related to a section $s(x)$ of $G \rightarrow P \backslash G$ defined by

$$x \mapsto s(x) = \begin{pmatrix} 1 & -ix \\ 0 & 1 \end{pmatrix} \quad (2)$$

where each entry is a 2×2 matrix. By using standard homogeneous space techniques one can easily verify that $SL(4, \mathbb{C})$ gives rise to the usual conformal group transformations of Minkowski space.

The non-compact double fibration is obtained from the compact one, $(P_2 \backslash G \leftarrow P_{12} \backslash G \rightarrow P_1 \backslash G)$, by replacing $P_1 \backslash G$ by \mathbb{M} and tracing this around the diagram using the projections:

$$\begin{array}{ccc} & \pi_1^{-1}(\mathbb{M}) := \mathbb{F}_{12} = \mathbb{M} \times \mathbb{CP}^1 & \\ \swarrow \pi_2 & & \searrow \pi_1 \\ \pi_2 \circ \pi_1^{-1}(\mathbb{M}) := \mathbb{P} = \mathbb{CP}^3 \backslash \mathbb{CP}^1 & \Longleftrightarrow & \mathbb{M} \end{array}$$

3 Supersymmetry

It is straightforward to generalise the foregoing to the supersymmetric case. N -extended super twistor space \mathbb{T}_N is $\mathbb{C}^{4|N}$, the complex super vector space with four even and N odd dimensions; it is a representation space of the complex superconformal group $SL(4|N; \mathbb{C})$. Starting from \mathbb{T}_N we can construct a large number of flag supermanifolds [6] by looking at the parabolic subsupergroups of $G = SL(4|N; \mathbb{C})$.

One of the simplest examples is $N = 1$ (compactified) super Minkowski space, $\widetilde{\mathbb{M}}$ (i.e. the body is compact), which corresponds to the subgroup of $SL(4|1; \mathbb{C})$ consisting of matrices

of the form

$$\left(\begin{array}{cccc|c} \times & \times & & & \\ \times & \times & & & \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \hline \times & \times & & & \times \end{array} \right)$$

We identify non-compact super Minkowski space with an open set in $\widetilde{\mathbb{M}}$ in a similar fashion to the bosonic case. In local coordinates $z = (x, \theta, \varphi)$ we have

$$z \mapsto s(z) = \left(\begin{array}{cc|c} 1 & -iX & -i\theta \\ 0 & 1 & 0 \\ \hline 0 & -i\varphi & 1 \end{array} \right) \quad (3)$$

where $X^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} - \frac{i}{2}\theta^\alpha\varphi^{\dot{\alpha}}$. Left and right chiral superspaces, \mathbb{M}_L (\mathbb{M}_R) are associated with the subgroups of matrices of the forms (for $N = 1$):

$$\left(\begin{array}{cccc|c} \times & \times & & & \\ \times & \times & & & \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \hline \times & \times & \times & \times & \times \end{array} \right) \quad \text{and} \quad \left(\begin{array}{cccc|c} \times & \times & & & \times \\ \times & \times & & & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \hline \times & \times & & & \times \end{array} \right)$$

so that we have the double fibration $\mathbb{M}_L \leftarrow \mathbb{M} \rightarrow \mathbb{M}_R$ [6]. This is in fact an exceptional case in the sense that the space of interest, \mathbb{M} , is at the “top” of the diagram rather than at the bottom right. The above can be extended in an obvious way to N -extended super Minkowski space and the corresponding chiral superspaces. We shall denote N -extended super Minkowski space by \mathbb{M}_N , or simply by \mathbb{M} .

Further examples of superspaces which have been used in field theory fall into two classes:

- Super twistor spaces [7]: these have bodies which are ordinary twistor spaces, i.e. one of the 7 spaces of the previous section.
- harmonic superspaces [8, 9]: these have bodies of the form Minkowski space \times internal flag space.

The main example of supertwistor theory is given by the double fibration [10, 11] $\mathbb{L} \leftarrow \mathbb{G} \rightarrow \mathbb{M}$ where the subgroups defining \mathbb{G} and \mathbb{L} consist of matrices of the forms

$$\left(\begin{array}{cccc|ccc} \times & & & & & & \\ \times & \times & & & & & \\ \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \hline \times & \times & & & \times & \times & \times \\ \times & \times & & & \times & \times & \times \\ \times & \times & & & \times & \times & \times \end{array} \right) \quad \text{and} \quad \left(\begin{array}{cccc|ccc} \times & & & & & & \\ \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \hline \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & & \times & \times & \times \end{array} \right)$$

respectively (illustrated for $N = 3$). The space \mathbb{L} , sometimes called super ambitwistor space, is the space of complex super light-like lines in \mathbb{M} . In the real case this is replaced by

$$\begin{array}{ccc}
 & M \times \mathbb{CP}^1 & \\
 \pi_2 \swarrow & & \searrow \pi_1 \\
 L & \Longleftrightarrow & M
 \end{array}$$

The space $M \times \mathbb{CP}^1$ is called light-cone harmonic superspace [12], while L , the space of real super light-like lines in super Minkowski space, M , is a real subspace of dimension $(5|2N)$ of projective supertwistor space.

Harmonic superspaces [8, 13] are classified by a pair of integers p, q where $p + q \leq N$ and we suppose that $p, q \geq 1$.

The appropriate subgroups for this case consist of matrices of the form

$$\left(\begin{array}{cccc|cccccccc}
 \times & \times & & & \bullet & \bullet & \bullet & & & & \\
 \times & \times & & & \bullet & \bullet & \bullet & & & & \\
 \times & \times & \times & \times & \times & . & . & . & . & . & \times \\
 \times & \times & \times & \times & \times & . & . & . & . & . & \times \\
 \hline
 \times & \times & & & \times & . & \times & & & & \\
 . & . & & & . & . & . & & & & \\
 . & . & & & \times & . & \times & & & & \\
 . & . & & & \times & . & . & . & \times & & \\
 . & . & & & . & . & . & . & . & & \\
 . & . & & & \times & . & . & . & \times & & \\
 \times & \times & \bullet & \bullet & \times & . & . & . & . & . & \times \\
 \times & \times & \bullet & \bullet & . & . & . & . & . & . & . \\
 \times & \times & \bullet & \bullet & \times & . & . & . & . & . & \times
 \end{array} \right) \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} p \quad \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} q$$

The space determined by the subgroup with no bullets is (compactified) (N, p, q) harmonic superspace, $\widetilde{\mathbb{M}}_N(p, q)$, whereas the space determined by the subgroup including the bullets is called (compactified) analytic (p, q) superspace, $\widetilde{\mathbb{M}}_{NA}(p, q)$. These two spaces form a double fibration with super Minkowski space: $\widetilde{\mathbb{M}}_{NA}(p, q) \leftarrow \widetilde{\mathbb{M}}_N(p, q) \rightarrow \widetilde{\mathbb{M}}_N$. In the non-compact case $\mathbb{M}_N(p, q) = \mathbb{M}_N \times \mathbb{F}_{p, N-q}(N)$ where the internal flag space (\mathbb{F} for short in the following) is represented by the bottom right part of the above diagram.

The correspondence is between points of \mathbb{M} and copies of \mathbb{F} in \mathbb{M}_{NA} and between points of analytic space and $\triangle(p, q)$ planes in \mathbb{M} . The latter are planes of dimension $(0|2(p+q))$ which have $2p$ tangent vectors with undotted indices (of the form $X^{\alpha i}, i = 1, \dots, N$) and $2q$ tangent vectors with dotted indices (of the form $Y_i^{\dot{\alpha}}$) such that $X^{\alpha i} Y_i^{\dot{\alpha}} = 0$ for any

such pair of vectors. A basis for the tangent spaces to any such plane is given by the derivatives

$$D_{\alpha r} = u_r^i D_{\alpha i} \quad D_{\dot{\alpha}}^{r'} = v_i^{r'} D_{\dot{\alpha}}^i \quad (4)$$

where $r = 1, \dots, p$; $r' = (N - q + 1) \dots N$, the D 's are the usual superspace covariant derivatives and where the matrices u, v have maximal rank and satisfy

$$u_r^i v_i^{r'} = 0. \quad (5)$$

It is not difficult to see that this equation defines the flag manifold $\mathbb{F}_{p, N-q}(N)$. A field, A , on $\mathbb{M}_N(p, q)$ is called G -analytic if it satisfies the generalised chirality constraints

$$D_{\alpha r} A = D_{\dot{\alpha}}^{r'} A = 0. \quad (6)$$

Such a field is equivalent to a field on analytic superspace.

4 Reality

The prototype twistor double fibration collapses to a single fibration of the form $\mathbb{R}^4 \times \mathbb{CP}^1 \rightarrow \mathbb{R}^4$ when Minkowski space is replaced by Euclidean space, \mathbb{R}^4 ; this is the starting point relevant to the study of self-dual Yang-Mills theory in Euclidean space. In a similar manner, it is often convenient to consider only single fibrations in the case of the chiral and harmonic supergeometries appropriate for real super Minkowski space. However, whereas in the case of Euclidean twistors the twistor space $Z = \mathbb{R}^4 \times \mathbb{CP}^1$ is naturally a complex space (it is “almost” \mathbb{CP}^3), in the supersymmetric case it is necessary to generalise the notion of a complex structure to that of a CR structure [14].

Let M be a real $(2n + m)$ -dimensional (super)manifold where $n, m \in \mathbb{Z}$ (\mathbb{Z}^2 in the super-case), and let T_c be the complexified tangent bundle. A CR structure (of rank n) on M is an n -dimensional sub-bundle of K of T_c such that

- $K_p \cap \bar{K}_p = 0, \forall p \in M$
- $X, Y \in \Gamma(K) \Rightarrow [X, Y] \in \Gamma(K)$

where \bar{K} denotes the complex conjugate of K , $K_p(\bar{K}_p)$ the fibre of $K(\bar{K})$ at $p \in M$, and $\Gamma(\cdot)$ denotes the space of sections of a bundle. On a CR (super)manifold there is a generalisation of the Dolbeault operator $\bar{\partial}$ on a complex manifold which is denoted by $\bar{\partial}_K$ and defined on scalars by $\bar{\partial}_K f = \pi \circ df$ where π denotes the projection: one-forms \rightarrow sections of \bar{K}^* , the dual space of \bar{K} . A CR analytic function is one which satisfies $\bar{\partial}_K f = 0$. A simple example of the above is given by real $N = 1$ superspace M which is a CR supermanifold of rank $(0|2)$ with $\bar{\partial}_K$ being simply the operator $\bar{D}_{\dot{\alpha}}$.

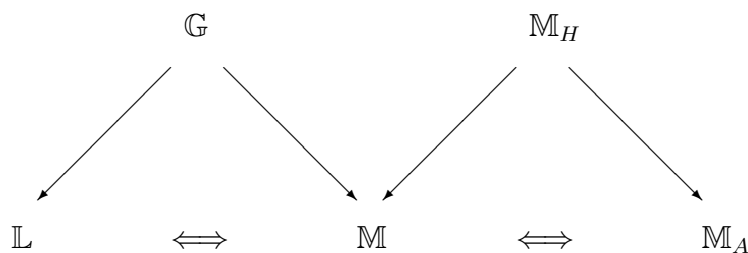
Real harmonic superspace $M_N(p, q)$ is $M_N \times \mathbb{F}$ where \mathbb{F} is the same internal complex flag manifold as in the previous section but which can also be thought of as the coset space $S((U(p) \times U(N - p - q) \times U(q)) \backslash SU(N)$. (Thus a field on $M_N(p, q)$ can be expanded

in $SU(N)$ harmonics on \mathbb{F} with coefficients which are ordinary superfields, whence the nomenclature). It is a CR supermanifold of rank $(\dim_c \mathbb{F}|2(p+q))$. The components of the CR operator $\bar{\partial}_K$ are $\{\bar{\partial}_F, D_{\alpha r}, D_{\dot{\alpha}}^{r'}\}$ where $\bar{\partial}_F$ is the usual Dolbeault operator on \mathbb{F} . In addition one can have G -analytic fields which are annihilated by the odd derivatives $D_{\alpha r}, D_{\dot{\alpha}}^{r'}$, but which need not be analytic on \mathbb{F} .

5 Applications

1) Massless on-shell supermultiplets can in many cases be described by CR analytic superfields in appropriate harmonic superspaces [13]. For example, $N = 8$ linearised SG is described by a single component superfield, W , on $M_8(4,4)$ superspace. The $N = 8$ linearised three-loop counterterm [15, 16] can be written in the manifestly supersymmetric form $\int d\mu W^4$ where $d\mu$ is the measure for G -analytic fields on $M_8(4,4)$, (generalised chiral measure) [13].

2) SYM theories. For $N = 2, 3$ the superspace constraints of SYM [17] may be described by applying the Ward construction to either the super twistor [10] or harmonic [8, 18] double fibrations (complex case); that is, we have the double double fibration:



where \mathbb{M}_H (\mathbb{M}_A) denotes the relevant harmonic (analytic) superspace. In the real case, it appears that only the harmonic formalism can be used to write actions [8, 18]. For $N = 1$ the supertwistor diagram is still relevant, but the chiral double fibration can also be used.

3) Superconformal geometry. In complex superspaces, superconformal transformations can be defined as those transformations of the correspondence space that leave the double fibration invariant [19]. In the case of real chiral and harmonic superspaces, this reduces to the preservation of the appropriate CR structures, although there are some subtleties [20, 19]. Deformations of these CR structures (for $N \leq 4$) determine off-shell field representations of conformal supergravity (CSG).

4). Non-linear CSG. The CR structure of $N = 1$ superspace gives a global description of Ogievetski-Sokatchev supergeometry [21, 22]. For $N \geq 2$, the geometry of CSG [23, 24] resembles quaternionic geometry [25] in the non-supersymmetric case. In this latter case one has a complex twistor space, Z , which is a fibre bundle over the quaternionic manifold M with fibre \mathbb{CP}^1 , whereas in the harmonic superspace description of CSG, harmonic superspace is a CR supermanifold as well as being a fibre bundle which fibres over superspace M with fibre the flag manifold, \mathbb{F} [13].

- 5) Extended Poincaré SG. The basic constraints of on-shell extended supergravity theories are superconformal [26] and can again be obtained from the perspective of the CR supergeometry of harmonic superspace, at least for $(p, q) = (1, 1)$ [13].
- 6) Lower dimensional spacetimes. For spacetime dimensions $d = 1, 2, 3$, constructions similar to those given here can be carried out using the appropriate superconformal groups $OSp(N|1)$ and $OSp(N|2)$ [27, 28], see also [29].
- 7) Higher dimensional spacetimes. With the exception of $d = 6$ the superconformal approach is not so useful in dimensions greater than 4. Instead twistors make their appearance via the interpretation of the d -dimensional Lorentz group $SO(1, d - 1)$ as the conformal group of $(d - 2)$ -dimensional Euclidean space. When combined with supersymmetry this has applications via lightlike integrability [30] and pure spinors [31] to higher dimensional field theories as well as to supersymmetric particles [32] and extended objects [33].
- 8) Euclidean and $(2, 2)$ signatures. It is possible to impose reality on the formalism in other ways so as to construct superspaces associated with Euclidean space or with four-dimensional space with a $(2, 2)$ signature metric. In these situations, self-duality can be supersymmetrised and supertwistor techniques applied. See, for example references [34].
- 9) Quantum supersymmetry. Up to now the applications of the above formalism to quantum supersymmetry have been to perturbation theory using off-shell superfield formalisms (where applicable) and superspace Feynman diagrams. However, there are indications that there may be direct applications to non-perturbative quantum field theory. Work on this is currently in progress [35].

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